HEAT TRANSFER AND DRAG IN THE LAMINAR FLOW OF A GAS WITH VARIABLE PROPERTIES THROUGH AN ANNULAR CHANNEL

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The drag and heat transfer problem has been solved numerically, in the boundary-layer approximation, for the laminar flow of a gas with variable properties through an annular channel.

Much attention is nowadays paid to the problems of heat transfer and drag in the laminar flow of a gas with variable properties through annular channels. This interest has been stimulated largely by the fact that heat exchangers in many modern engineering applications operate at high temperatures. Inasmuch as it is necessary here to take into account that the physical properties of a gas are functions of the temperature, it is not possible to solve these problems by classical methods. For this reason, numerical methods are used instead. Several references can be cited where hydrodynamic and heat transfer problems have been solved by numerical methods for the laminar flow of a gas with variable properties through circular pipes [1, 2, 7]. The authors know of no reference, however, where the drag and heat transfer problem has been solved for the laminar flow of a gas with variable properties through an annular channel.

Fundamental Equations. In order to describe the flow through pipes, one often uses the boundary-layer equations [1, 2, 5]. Here the problem of flow through an annular channel will be solved in the boundary-layer approximation. In the boundary-layer approximation, as is well known, the terms representing molecular heat and axial momentum transfer are omitted from both the energy and the flow equation; it is also assumed that the radial pressure gradient is small in comparison with the axial pressure gradient. While analyzing the results, we will evaluate these omitted terms.

If one disregards the terms accounting for energy dissipation and assumes that there are no internal sources of heat in the stream, then the dimensionless boundary-layer equations for the flow of a gas with variable physical properties in a vertical annual channel will be

$$\frac{\partial(\rho u)}{\partial x}+\frac{1}{r}\cdot\frac{\partial(r\rho v)}{\partial r}=0, \tag{1}$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial r}=-\frac{dp}{dx}\pm\frac{\mathrm{Ga}_0}{4\mathrm{Re}_0}\,\mathrm{Pr}_0\left(1-\frac{1}{\Theta}\right)+\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\mu\,\mathrm{Pr}_0\frac{\partial u}{\partial r}\right), \tag{2}$$

$$\rho u\frac{\partial H}{\partial x}+\rho v\frac{\partial H}{\partial r}=\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\lambda}{c_p}\cdot\frac{\partial H}{\partial r}\right), \tag{3}$$

where

$$x=\frac{X/R_2-R_1}{\dfrac{U_0\mathrm{P}_0(R_2-R_1)\,\mathrm{Pr}_0}{\mathrm{M}_0}};\quad u=\frac{U}{U_0};\quad r=\frac{R}{R_2-R_1};$$

$$\rho=\frac{\mathrm{P}}{\mathrm{P}_0};\quad v=\frac{v}{u_0}\cdot\frac{\mathrm{Re}_0\mathrm{Pr}_0}{2};\quad c_p=\frac{C_p}{C_{p0}};$$

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$$\mu = \frac{M}{M_0}; \quad \Theta = \frac{T}{T_0}; \quad H = \frac{h}{h_0}; \quad p = \frac{P - P_0}{P_0 U_0^2};$$

$$\mathrm{Re}_0 = \frac{2U_0(R_2 - R_1)P_0}{M_0}; \quad \mathrm{Pr}_0 = \frac{\nu_0}{a_0};$$

and $Ga_0 = 8gP_0^2(R_2-R_1)^3/M_0^2$ is the Galileo number.

If one considers an ideal gas, then one can add to systems (1)-(3) also the equation of state

$$P = PR_{gas}T. \tag{4}$$

The boundary conditions in dimensionless form are defined as follows:

$$\begin{aligned} &x = 0, \quad r_1 \leqslant r \leqslant r_2 : u = 1, \quad v = 0, \quad H = 1, \quad \Theta = 1; \\ &x > 0, \quad r = r_1 = \frac{N}{1-N} : u = v = 0, \quad \Theta = \Theta_1, \quad H = H_1; \\ &x > 0, \quad r = r_2 = \frac{1}{1-N} : u = v = 0, \quad \Theta = \Theta_2, \quad H = H_2. \end{aligned} \tag{5}$$

Thus, the problem is solved for boundary conditions of the first kind at the walls of an annular channel and for uniform velocity and temperature profiles at the channel entrance.

Calculation Procedure. Equations (1)-(4) with the boundary conditions (5) were solved numerically on a Minsk-22 digital computer by the sweep method according to the Patankar–Spalding procedure [3, 4]. The main advantage of this procedure is that iterations can be avoided. This considerably shortens the machine time required for solving the problem. As the transverse coordinate we used the dimensionless function ω:

$$\omega = \frac{\psi - \psi_I}{\psi_E - \psi_I}, \tag{6}$$

where ψ_I and ψ_E are the values of the flow function at the inside and at the outside boundary of the analyzed region.

With the longitudinal coordinate (x) fixed, the flow function can be determined from the continuity equation:

$$d\psi = \rho u r dr. \tag{7}$$

It may be assumed that at the inside wall of an annular channel the flow function $\psi_I = 0$. Then

$$\omega = \psi/\psi_E. \tag{8}$$

Systems (1)-(4) become in the x–ω coordinates (modified Mises coordinates):

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \omega}\left(\frac{\mu \rho u r^2 \mathrm{Pr}_0}{\psi_E^2}\frac{\partial u}{\partial \omega}\right) - \frac{\frac{dp}{dx} \mp \frac{\mathrm{Ga}_0 \mathrm{Pr}_0}{4\mathrm{Re}_0}\left(1 - \frac{1}{\Theta}\right)}{\rho u}; \tag{9}$$

$$\frac{\partial H}{\partial x} = \frac{\partial}{\partial \omega}\left(\frac{\lambda \rho u r^2}{c_p \psi_E^2} \cdot \frac{\partial H}{\partial \omega}\right). \tag{10}$$

The flow function at the outside wall will be determined by integrating (7):

$$\psi_E = \int_0^1 \rho u r dy, \tag{11}$$

where y is the dimensionless distance measured from the inside surface and

$$r = r_1 + y = \frac{N}{1-N} + y, \quad y = \frac{Y}{R_2 - R_1}. \tag{12}$$

At the inside surface of an annular channel $y_1 = 0$ ($\omega = 0$), at the outside surface $y_2 = 1$ ($\omega = 1$). Without any heat sources inside the channel and without any drain of mass, ψ_E = const. One can calculate ψ_E on the basis of (11), considering that at the channel entrance $\rho = 1$, $u = 1$:

$$\psi_E = \frac{1+N}{2(1-N)}. \tag{13}$$

The relation between the dimensionless flow function and the y coordinate will be established from the following expression:

$$\omega = \frac{\psi}{\psi_E} = \frac{\int_0^y \rho u r dy}{\int_0^1 \rho u r dy} = \frac{\int_0^y \rho u r dy}{\psi_E}. \tag{14}$$

The main difficulty in solving the problem inside a channel is to determine the local pressure gradient dp/dx along the channel axis. Spalding has developed a general and rather simple procedure for determining dp/dx by which iterations can be avoided. The pressure gradient is determined from the equilibrium conservation momentum, which for an annular channel can be written as follows:

$$\frac{d}{dX}\int_{R_1}^{R_2} 2\pi R P U^2 dR + \pi (R_2^2 - R_1^2)\frac{dP}{dX} + 2\pi (r_1\tau_1 + r_2\tau_2) = 0. \tag{15}$$

We transform (14) and rewrite it in dimensionless form:

$$\frac{dp}{dx} = -\frac{2\mathrm{Pr}_0}{1+N}\left[N\mu_1\left(\frac{\partial u}{\partial y}\right)_1 - \mu_2\left(\frac{\partial u}{\partial y}\right)_2\right] - \frac{2(1-N)}{1+N}\frac{d}{dx}\int_{r_1}^{r_2}\rho u^2 r dr. \tag{16}$$

In order to avoid iterations in calculating the pressure gradient, we introduce a correction (in accordance with Spalding's concepts [5]) to account for the possibility of the stream either not filling the entire channel section or spilling over the outer boundary – if dp/dx has not been determined accurately. The magnitude of this correction depends on the difference between the true section area and the calculated section area covered by the stream. In the final form, the expression for the pressure gradient becomes

$$\frac{dp}{dx} = -\frac{2\mathrm{Pr}_0}{1+N}\left[N\mu_1\left(\frac{\partial u}{\partial y}\right)_1 - \mu_2\left(\frac{\partial u}{\partial y}\right)_2\right] - \frac{1}{\psi_E}\frac{k_d - k_u}{x_d - x_u} + \frac{mu(A_d - A_u)}{A_u^2(x_d - x_u)}, \tag{17}$$

with $m = (1-N)/(1+N)$ denoting the dimensionless mass flow rate along the channel, $A_d = 1/(1-N)^2 - N^2/(1-N)^2$ denoting the dimensionless true area of a channel cross section, $A_u = (N/(1-N) + y_{27})^2 - N^2/(1-N)^2$ denoting the calculated dimensionless area, and $k = \sum_{i=1}^{27} [\rho_i u_i^2 (N/(1-N) + y_i) + \rho_{i-1} u_{i-1}^2 (N/(1-N) + y_{i-1})](y_i - y_{i-1})/2$ denoting the integral sum which determines the kinetic energy of the stream (for the purpose of calculations, the distance from the inside to the outside wall was subdivided into 27 steps).

Furthermore, the "historical weighing" technique [5] was used in the determination of dp/dx. Calculations were made here for hydrogen treated as an ideal gas. The effect of pressure changes on the properties of the gas was disregarded.

The Sutherland formula

$$\mu = \frac{1 + \frac{84.4}{T_0}}{\Theta + \frac{84.4}{T_0}}\Theta^{3/2} \tag{18}$$

was used for calculating the dynamic viscosity as a function of the temperature. The Jacob formula [6] was used for expressing the thermal conductivity as a function of the temperature:

$$\lambda = \frac{\left(\frac{1}{T_0} + 247\cdot 10^{-6}\Theta\right)(T_0 + 83)\sqrt{\Theta}}{\left(T_0 + \frac{83}{\Theta}\right)\left(\frac{1}{T_0} + 247\cdot 10^{-6}\right)}. \tag{19}$$

The specific heat was approximated by a linear function of the temperature:

$$c_p = 1 + 98.3\cdot 10^{-6}(T_0\Theta - 273.2). \tag{20}$$

The dimensionless enthalpy was determined from the relation

$$H - 1 = \int_0^\Theta c_p d\Theta. \tag{21}$$

The results of heat transfer calculations were organized in terms of the Nusselt number based on the mean-mixed temperature

$$\mathrm{Nu}_1 = \frac{2q_1}{\lambda_m(\Theta_1 - \Theta_m)}, \tag{22}$$

where

$$q_1 = -\lambda_1 \left(\frac{\partial\Theta}{\partial y}\right)_1. \tag{23}$$

The mean-mixed temperature Θ_m was determined from the mean-mixed enthalpy H_m:

$$H_m = \frac{\int_0^1 \rho u H r dy}{\int_0^1 \rho u r dy} = \frac{\int_0^1 \rho u H r dy}{\psi_E}. \tag{24}$$

The hydraulic drag coefficient was calculated according to the formula

$$\zeta = -\frac{2d_e}{P_m U_m} \cdot \frac{dP}{dX} \tag{25}$$

or in the dimensionless form:

$$\zeta \mathrm{Re}_m = -\frac{8}{u_m \mu_m \mathrm{Pr}_0} \cdot \frac{dp}{dx}. \tag{26}$$

The calculations were made for a hydrogen temperature $T_0 = 293.2°K$ at the channel entrance. The temperature of the inside surface of the annular channel was 440, 1273, and 1773°K ($\Theta_1 = 1.5$, 4.342, and 6.047, respectively) and the temperature of its outside surface was 323.2°K ($\Theta_2 = 1.102$).

Analysis of the Calculated Results. The calculated local values of the Nusselt number and of the hydraulic drag coefficient, as well as the development of longitudinal temperature and velocity fields are shown in Figs. 1-4. The effect of changes in the properties on the Nusselt number is limited to the initial heating range, while no appreciable change in the referred length of this range is noted. The local values of the Nusselt number beyond the initial heating range are close to those for a gas with constant properties (Fig. 1). Within the initial range the Nusselt number is somewhat higher than for a flow with constant properties at very small referred lengths ($x < 0.01$) and somewhat lower than for a flow with constant properties at larger referred lengths ($x > 0.1$). As the dimensionless temperature at the inside wall rises, the Nusselt number for small referred lengths increases. It increases also as the ratio of inside radius to outside radius of an annular channel decreases.

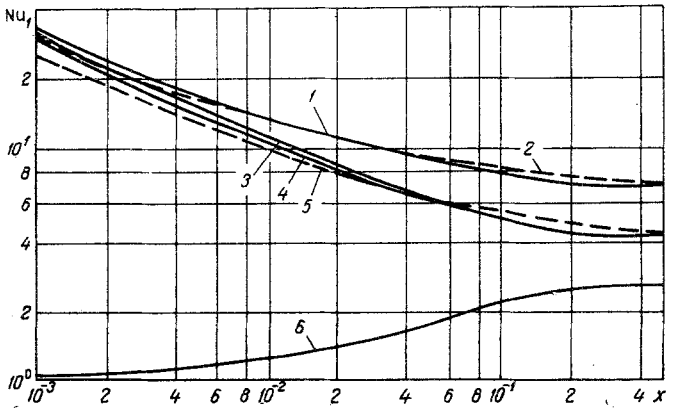

Fig. 1. Variation of the Nusselt number at the inside surface of an annular channel: 1) N = 0.2, $\Theta_1 = 4.342$; 2) N = 0.2 and constant properties – dashed line; 3) N = 0.8, $\Theta_1 = 6.047$; 4) N = 0.8, $\Theta_1 = 4.342$; 5) N = 0.8 and constant properties – dashed line; 6) Θ_m for N = 0.8, $\Theta_1 = 4.342$.

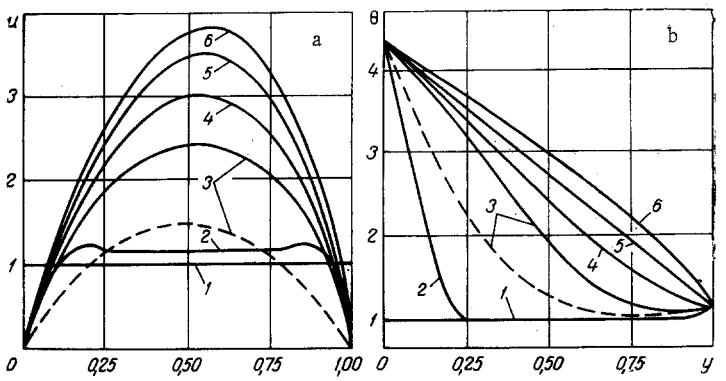

Fig. 2. Development of temperature profiles (a) and velocity profiles (b): 1) x = 0; 2) 0.0013; 3) 0.04 (dashed line for constant properties); 4) 0.082; 5) 0.146; 6) 0.49.

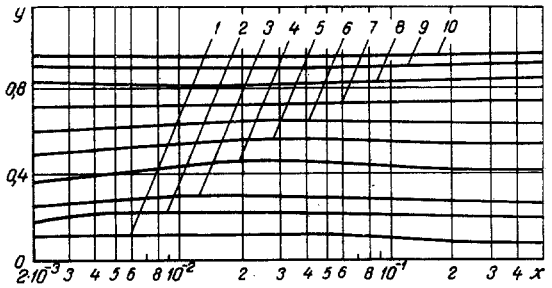

Fig. 3. Change in streamlines along a channel: 1) $\omega = \psi/\psi_E = 0.0128$; 2) 0.0512; 3) 0.1; 4) 0.25; 5) 0.4; 6) 0.55; 7) 0.7; 8) 0.85; 9) 0.95; 10) 0.99.

During a flow with variable properties the temperature gradient at the inside hot surface is smaller than during a flow with constant properties (Fig. 2a). This decrease can be explained by the more intensive heating of the stream as a result of the sharp decrease in its density within the boundary layer. The streamlines deflect from the inside wall toward the main stream, i.e., an "overflow" of gas into the main stream takes place (Fig. 3). In this study, with the particular system of coordinates (x–ω), the problem of specially determining the transverse velocity component was not considered. The data in Fig. 3, which represent the change in the streamlines along the channel, indicate directly the effect of the transverse velocity component and of the gas "overflow" into the main stream. The streamlines are not deformed beyond the initial range, i.e., the radial velocity component asymptotically approaches zero.

The velocity gradients at the walls are much larger in a flow of a gas with variable properties (Fig. 2b), which brings about a higher dynamic viscosity and also an appreciably higher friction drag. For this reason, the values of the hydraulic drag coefficient are very different here than in the case of a flow with constant properties. This difference increases as the wall temperature rises (Fig. 4).

The results obtained here are in qualitative agreement with the results of heat transfer calculations for the laminar flow of a gas with variable properties through a circular pipe [1, 2]. The physical aspects of this process, on the basis of which one can explain the variation of the Nusselt number for the laminar flow with variable properties through a circular pipe, have been thoroughly described in [1, 2]. The data in Figs. 1-4 lead one to conclude that the identical processes occur in the laminar flow of a gas with variable properties through an annular channel.

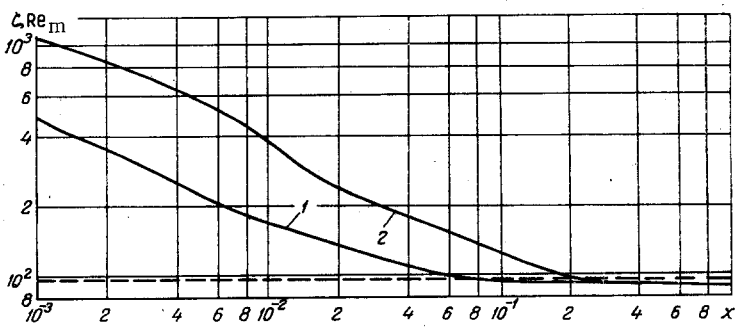

Fig. 4. Variation of the hydraulic drag coefficient along an annular channel (N = 0.8): 1) constant properties; 2) Θ_1 = 4.342 (dashed line for isothermal steady flow).

On the basis of these results, we have evaluated the terms which account for the molecular heat and momentum transfer in the axial direction.

The ratio of molecular transfer to convective transfer along the axis is determined in dimensionless coordinates from the following relations:

$$\frac{4\mathrm{Pr}_0}{\mathrm{Pe}_0^2}\cdot\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)\left(\rho u\frac{\partial u}{\partial x}\right)^{-1}, \tag{27}$$

$$\frac{4}{\mathrm{Pe}_0^2}\cdot\frac{\partial}{\partial x}\left(\frac{\lambda}{c_p}\cdot\frac{\partial H}{\partial x}\right)\left(\rho u\frac{\partial H}{\partial x}\right)^{-1}. \tag{28}$$

An evaluation of the quantities in expressions (27) and (28) for x = 0.001, N = 0.8, and $\Theta_1 = 4.342$ has yielded for the relative molecular momentum transfer along the channel axis $2.744\cdot 10^3/\mathrm{Pe}_0^2$, and molecular heat transfer $4\cdot 10^3/\mathrm{Pe}_0^2$. If one assumes that this fraction cannot be greater than 2%, then $\mathrm{Pe}_0 = 450$ (referred to the molecular heat transfer along the axis). The minimum value of x at which the calculated results begin to become valid will be found from the relation

$$x = \frac{4\mathrm{Pe}_x}{\mathrm{Pe}_0^2}. \tag{29}$$

The radial pressure gradient has not been evaluated, because it is usually negligibly small as compared to the axial pressure gradient, and the minimum allowable value of the Peclet number Pe_0 is determined mainly by the relative molecular heat and momentum transfer in the axial direction.

NOTATION

X	is the axial coordinate along the stream;
R	is the radial coordinate;
M	is the dynamic viscosity;
P	is the density;
R_{gas}	is the gas constant;
$N = R_1/R_2$	is the parameter characterizing the geometry of an annular channel;
$\mathrm{Pe}_X = U_0 X a_0^{-1}$	is the Peclet number along the axial coordinate;
Re_m	is the Reynolds number with respect to mean-mixed temperature.

Other symbols are conventional.

Subscripts

0	denotes the channel entrance;
1	denotes the inside wall of the channel;
2	denotes the outside wall of the channel;
m	denotes the mean-mixed value;
u	denotes preceding in the flow;
d	denotes succeeding in the flow.

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